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Zhurnal Eksperimentalnoy i Teoreticheskoy Fiziki

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An approximate method is given for the calculation of the penetration of a magnetic field into a superconductor of an arbitrary shape.

Let us examine a superconductor on which an external heterogeneous magnetic field is imposed. It is known that an increase of the dimensions of the superconductor increases the accuracy with which the magnetic field, surrounding the superconductor, corresponds with the magnetic field which surrounds the ideal diamagnetic under the same conditions. The diamagnetic corresponds in dimensions and shape with the investigated superconductor. With the decrease in dimensions, this correspondence is lost, however. We would like to consider here the following conditions: If the superconductor is an arbitrarily curved plate of thickness d, the penetration of the magnetic field into it can be calculated accurately from the theory of a superconductor as an ideal diamagnetic, even if $\delta \sim d$, where δ is the parameter which characterizes the depth of penetration of the field into the superconductor (Ref. 1). Taking the ideal diamagnetic as the zeroth approximation to the superconductor ($\delta = 0$), we do not obtain any penetration into the body. However, the first approximation, which is determined below, gives us a law of field penetration into a plate which appears to be very accurate, even if $\delta \sim d$.

This approximation does not lead to a correct result only if the plate is bent in such a manner that the radius of curvature R, which, in general, is different at each point on the plate surfaces, becomes comparable to the depth of

[1] *Journal of Experimental and Theoretical Physics.*

penetration, δ. This can be confirmed by a calculation of the field penetration into a diamagnetic or, for the superconductor, using the method of perturbations.[2]

A phenomenalogical equation which describes superconductivity has the following form (Ref. 3):

$$\delta^2 \text{ rot } \mathbf{j_s} = -\mathbf{H}, \quad \text{rot } \mathbf{H} = \mathbf{j_s} \tag{1}$$

For the sake of convenience, $\mathbf{H}$ does not define the magnetic field itself, but the magnitude, which is $c/4\pi$ times larger, and $\mathbf{j_s}$ is the superconductive current.

The solution to the system of equations (1) by the method of perturbations actually defines the following. Let us assume that we want to calculate the penetration of the magnetic field into a superconductive plate which has a random thickness and is bent in a random manner. For the sake of simplicity, we will assume that both surfaces of that plate permit the introduction of a curvilinear orthogonal coordinate system (α, β, γ), where the surfaces of the superconductor are actually coordinate surfaces, defined by $\gamma = \gamma_+$, and $\gamma = \gamma_-$. Then, we will look for a solution to the system of equations (1) in the form of:

$$\begin{aligned} \mathbf{H} &= (\mathbf{A}_0 + \delta \mathbf{A}_1 + \delta^2 \mathbf{A}_2 + \cdots)\, e^{\psi(\gamma')} \\ \mathbf{j_s} &= (\delta^{-1} \mathbf{B}_{-1} + \mathbf{B}_0 + \delta \mathbf{B}_1 + \delta^2 \mathbf{B}_2 + \cdots)\, e^{\psi(\gamma')} \end{aligned} \tag{2}$$

where $\gamma' = \gamma/\delta$, and $\mathbf{A}$ and $\mathbf{B}$ are functions only of α, β, and γ. We substitute equation (2) into equation (1) and collect the terms which have the same degree in δ. Because of the arbitrariness of δ, we can equate the corresponding coefficients to zero. By utilizing the fact that the resulting equations are not independent, it is possible to determine the function $\psi' = d\psi/d\gamma'$ and to make a number of simplifications. As a result we will find:

[2]Such a method for a problem solution was developed for the first time by S. M. Rytov and was applied in the calculation of the skin effect (Ref. 2).

$$j_{S\alpha} = e^{\psi_-} (\delta^{-1} B^-_{-1\alpha} + B^-_{0\alpha} + \delta B^-_{1\alpha} + \cdots) + e^{\psi_+} (\delta^{-1} B^+_{-1\alpha} + B^+_{0\alpha} + \delta B^+_{1\alpha} + \cdots)$$

$$j_{S\gamma} = e^{\psi_-} \left\{ h_\alpha h_\beta \left(\frac{\partial (B^-_{-1\alpha}/h_\beta)}{\partial \alpha} + \frac{\partial (B^-_{-1\beta}/h_\alpha)}{\partial \beta} \right) + \cdots \right\} + e^{\psi_+} \left\{ h_\alpha h_\beta \left(\frac{\partial (B^+_{-1\alpha}/h_\beta)}{\partial \alpha} + \frac{\partial (B^+_{-1\beta}/h_\beta)}{\partial \beta} \right) + \cdots \right\}$$

$$H_\alpha = e^{\psi_-} \left\{ -B^-_{-1\beta} + \delta \left[-B^-_{0\beta} + \frac{h_\gamma}{2} \frac{\partial \ln (h_\alpha/h_\beta)}{\partial \gamma} B^-_{-1\beta} \right] + \cdots \right\}$$

$$+ e^{\psi_+} \left\{ B^+_{-1\beta} + \delta \left[B^+_{0\beta} + \frac{h_\gamma}{2} \frac{\partial \ln (h_\alpha/h_\beta)}{\partial \gamma} B^+_{-1\beta} \right] + \cdots \right\} \tag{3}$$

$$H_\gamma = e^{\psi_-} \delta \left\{ h_\alpha h_\beta \left(\frac{\partial (B^-_{-1\alpha}/h_\alpha)}{\partial \beta} - \frac{\partial (B^+_{-1\beta}/h_\beta)}{\partial \alpha} \right) + \cdots \right\}$$

$$+ e^{\psi_+} \delta \left\{ h_\alpha h_\beta \left(\frac{\partial (B^+_{-1\alpha}/h_\alpha)}{\partial \beta} + \frac{\partial (B^+_{-1\beta}/h_\beta)}{\partial \alpha} \right) + \cdots \right\} \tag{4}$$

where $j_{S\beta}$ and H_β are obtained respectively from $j_{S\alpha}$ and H_α by a reciprocal exchange of indices α and β. Here, h_α, h_β, and h_γ are metric coefficients and

$$\psi_- = -\frac{1}{\delta} \int_{\gamma_-}^{\gamma} \frac{d\gamma}{h_\gamma}, \qquad \psi_+ = \frac{1}{\delta} \int_{\gamma_+}^{\gamma} \frac{d\gamma}{h_\gamma} \tag{5}$$

In order to obtain the final solution, the boundary conditions of the problem must be satisfied. Since we have assumed that the magnetic penetration into a superconductor equals unity, it follows that the corresponding boundary conditions are

$$^{e}\mathbf{H} = \mathbf{H}, \text{ when } \gamma = \gamma_+, \gamma_- \tag{6}$$

where $^{e}\mathbf{H}$ is the field outside the superconductor. After expressing $^{e}\mathbf{H}$ as a power series in δ and substituting for $\mathbf{H}$ in equation (6) the expressions of equation (4), then the terms containing the same power δ are collected. By equating the corresponding coefficients to zero, we obtain

$$^{e}H_{0\gamma} = 0 \text{ when } \gamma = \gamma_+, \gamma_-, \tag{7}$$

and furthermore, a number of correlations which make it possible to express $B^{\pm}_{0\alpha}$, $B^{\pm}_{0\beta}$, $B^{\pm}_{0\gamma}$ in terms of $^{e}\mathbf{H}$ at the boundary of the superconductor. By limiting ourselves to the first approximation only, we obtain the following expression for the superconductive current and the magnetic field inside the superconductor:

$$j_{S\alpha} = \frac{e^{\psi_-}}{e^{2\psi}-1}\,\frac{1}{\delta}\left\{\frac{^{e}H^{-}_{0\beta}}{\sqrt{(h_\alpha h_\beta)_-}} - e^{\psi}\,\frac{^{e}H^{+}_{0\beta}}{\sqrt{(h_\alpha h_\beta)_+}}\right\}\sqrt{h_\alpha h_\beta} \tag{8}$$

$$+\frac{e^{\psi_+}}{e^{2\psi}-1}\,\frac{1}{\delta}\left\{\frac{^{e}H^{-}_{0\beta}}{\sqrt{(h_\alpha h_\beta)_-}}\,e^{\psi} - \frac{^{e}H^{+}_{0\beta}}{\sqrt{(h_\alpha h_\beta)_+}}\right\}\sqrt{h_\alpha h_\beta}$$

$$j_{S\gamma} = h_\alpha h_\beta \frac{e^{\psi_-}}{e^{2\psi}-1}\left\{\frac{\partial}{\partial\alpha}\sqrt{\frac{h_\alpha}{h_\beta}}\left[\frac{{}^eH_{0\beta}^-}{\sqrt{(h_\alpha h_\beta)_-}} - e^{\psi}\frac{{}^eH_{0\beta}^+}{\sqrt{(h_\alpha h_\beta)_+}}\right]\right.$$

$$\left. + \frac{\partial}{\partial\beta}\sqrt{\frac{h_\alpha}{h_\beta}}\left[\frac{{}^eH_{0\alpha}^+}{\sqrt{(h_\alpha h_\beta)_+}}e^{\psi} - \frac{{}^eH_{0\alpha}^-}{\sqrt{(h_\alpha h_\beta)_-}}\right]\right\}$$

$$+ h_\alpha h_\beta \frac{e^{\psi_+}}{e^{2\psi}-1}\left\{\frac{\partial}{\partial\alpha}\sqrt{\frac{h_\alpha}{h_\beta}}\left[\frac{{}^eH_{0\beta}^-}{\sqrt{(h_\alpha h_\beta)_-}}e^{\psi} - \frac{{}^eH_{0\beta}^+}{\sqrt{(h_\alpha h_\beta)_+}}\right]\right.$$

$$\left. + \frac{\partial}{\partial\beta}\sqrt{\frac{h_\alpha}{h_\beta}}\left[\frac{{}^eH_{0\alpha}^+}{\sqrt{(h_\alpha h_\beta)_+}} - e^{\psi}\frac{{}^eH_{0\alpha}^-}{\sqrt{(h_\alpha h_\beta)_-}}\right]\right\} \tag{8a}$$

$$H_\alpha = \frac{e^{\psi_-}}{e^{2\psi}-1}\left\{\frac{{}^eH_{0\alpha}^-}{\sqrt{(h_\alpha h_\beta)_-}} - e^{\psi}\frac{{}^eH_{0\alpha}^+}{\sqrt{(h_\alpha h_\beta)_+}}\right\}\sqrt{h_\alpha h_\beta}$$

$$+ \frac{e^{\psi}}{e^{2\psi}-1}\left\{\frac{{}^eH_{0\alpha}^+}{\sqrt{(h_\alpha h_\beta)_+}} - e^{\psi}\frac{{}^eH_{0\alpha}^-}{\sqrt{(h_\alpha h_\beta)_-}}\right\}\sqrt{h_\alpha h_\beta} \tag{9}$$

$$H_\gamma = \delta h_\alpha h_\beta \frac{e^{\psi_-}}{e^{2\psi}-1}\left\{\frac{\partial}{\partial\beta}\sqrt{\frac{h_\beta}{h_\alpha}}\left[\frac{{}^eH_{0\beta}^-}{\sqrt{(h_\alpha h_\beta)_-}} - e^{\psi}\frac{{}^eH_{0\beta}^+}{\sqrt{(h_\alpha h_\beta)_+}}\right]\right.$$

$$\left. - \frac{\partial}{\partial\alpha}\sqrt{\frac{h_\alpha}{h_\beta}}\left[\frac{{}^eH_{0\alpha}^+}{\sqrt{(h_\alpha h_\beta)_+}}e^{\psi} - \frac{{}^eH_{0\alpha}^-}{\sqrt{(h_\alpha h_\beta)_-}}\right]\right\}$$

$$+ \delta h_\alpha h_\beta \frac{e^{\psi_+}}{e^{2\psi}-1}\left\{\frac{\partial}{\partial\beta}\sqrt{\frac{h_\beta}{h_\alpha}}\left[\frac{{}^eH_{0\beta}^-}{\sqrt{(h_\alpha h_\beta)_-}}e^{\psi} - \frac{{}^eH_{0\beta}^+}{\sqrt{(h_\alpha h_\beta)_+}}\right]\right.$$

$$-\frac{\partial}{\partial\alpha}\sqrt{\frac{h_\alpha}{h_\beta}}\left[\frac{{}^eH^+_{0\alpha}}{\sqrt{(h_\alpha h_\beta)_+}} - e^\psi\frac{{}^eH^-_{0\alpha}}{\sqrt{(h_\alpha h_\beta)_-}}\right]\Bigg\}$$

where

$$\psi = -\frac{1}{\delta}\int_{\gamma_-}^{\gamma_+}\frac{d\gamma}{h_\gamma} \tag{10}$$

(H_β and $-j_{S\beta}$ are obtained respectively from H_α and $j_{S\alpha}$ by a reciprocal exchange of the indices α and β).

Condition (7) itself represents a known boundary condition on the surface of an ideal diamagnetic. The constants ${}^eH^\pm_{0\alpha,\beta}$ (which are actually the functions of α and β, but *not* γ) are determined by comparing the solution of the following equation[3]

$$\Delta\phi = 0, \qquad \mathbf{H} = \nabla\phi \tag{11}$$

to the solution to equation (4) on the surface of the superconductor (condition 6). As can be seen from equation (4), it is possible to neglect the second approximation if

$$\frac{\delta h_\gamma}{2}\left|\frac{\partial \ln (h_\beta/h_\alpha)}{\partial\gamma}\right| << 1 \tag{12}$$

It is easily proved with the help of a number of examples that condition (12) defines the requirement that the depth of penetration be small in comparison with the radius of curvature of the superconductor surfaces. In the case of a flat parallel plate, which is placed in a heterogeneous magnetic field parallel to its surface, we obtain from equations (8) and (9),

[3]At infinity, solutions to equation (11) should give a heterogeneous magnetic field which has a definite direction.

$$H_\alpha = H_0 \frac{\mathrm{ch}(x/\delta)}{\mathrm{ch}(d/\delta)}, \quad H_\beta = H_\gamma = 0$$

$$j_{S\beta} = -\frac{c}{4\pi\delta}\frac{\mathrm{sh}(x/\delta)}{\mathrm{ch}(d/\delta)}, \quad j_{S\alpha} = j_{S\gamma} = 0 \tag{13}$$

That is; in this case we have a complete agreement with the accurate formulas (Ref. 1).

The complete agreement with the accurate formulas could have been due to the fact that, in the given case, the radius of a curvature of the surface was infinite, as well as to the fact that the normal component of the external magnetic field, with respect to the surface of the cylinder, is zero at infinity. However, in the example of a cylinder in a heterogeneous magnetic field perpendicular to its a axis, it is evident that the agreement is accomplished at any given value of the normal component of the field only if the inequality $R >> \delta$ holds.

It is easy to obtain formulae for the energy density of the magnetic field in a superconductor, and also formulae for $\mathbf{j}_S$ and $\mathbf{H}$ inside the superconductor with one surrounding surface (sphere, cylinder). However, we will not concern ourselves with this problem here.

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